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A note on the cyclical edge-connectivity of fullerene graphs

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A fullerene graph is a 3-regular (cubic) and 3-connected spherical graph that has exactly 12 pentagonal faces and other hexagonal faces. The cyclical edge-connectivity of a graph G is the maximum integer k such that G cannot be separated into two components, each containing a cycle, by deletion of fewer than k edges. Došlić proved that the cyclical edge-connectivity of every fullerene graph is equal to 5. By using Euler's formula, we give a simplified proof, mending a small oversight in Došlić's proof. Further, it is proved that the cyclical connectivity of every fullerene graph is also equal to 5.

KEY WORDS: fullerene graph, cyclical edge-connectivity, cyclical connectivity

1. Introduction

A *fullerene graph*, the molecular graph of a spherical carbon cluster, is a 3-regular (cubic) and 3-connected plane graph (or spherical map) that has exactly 12 faces of size 5 and other faces of size 6. In a series of articles [2–5, 12], it has been shown that fullerene graphs have certain structural properties related to matching theory, such as bicriticality, 2-extendability, etc.

A graph G is said to be *bicritical* if $G - u - v$ has a perfect matching for every pair of distinct vertices u and v. A connected graph G with at least $2k+2$ vertices is said to be *k-extendable* if it contains a matching of size k and every such matching is contained in a perfect matching. A graph G is *cyclically k-edge connected* if at least k edges must be removed to disconnect G into two components that each contains a cycle. *The cyclical edge-connectivity* of G, denoted by $c\lambda(G)$, is the maximum integer k such that G is cyclically k-edge connected [6, 7]. There is a relation between bicriticality and the cyclic edge-connectivity as follows (See Ex. 5.5.21 in [9]).

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Propositon 1. For a k-regular non-bipartite graph G ($k \geqslant 3$), if G is cyclically $(k+1)$-edge connected and has even number of vertices, then G is bicritical.

From proposition 1, Došlić [2] showed that all fullerene graphs are bicritical by giving the following lower bound on cyclical edge-connectivity of fullerene graphs:

Theorem 2. [2]. Let G be a fullerene graph. Then $c\lambda(G) \geqslant 4$.

From this result, the 2-extendability of all fullerene graphs was discovered by Zhang and Zhang [12] based on the result of Ref. 8: If G is a 3-connected cubic planar graph which is cyclically 4-edge connected and has no face of size 4, then G is 2-extendable.

Furthermore, combining Sashs' result that the cyclical edge-connectivity of every 2-edge connected cubic graph is no more than 5 [11], it was obtained that for every fullerene graph G, $c\lambda(G) \leqslant 5$.

Later, Došlić [5] determined the value of $c\lambda$ for all fullerene graphs, solving the problem proposed by Zhang and Zhang in [12]:

Theorem 3. [5]. Let G be a fullerene graph. Then $c\lambda(G) = 5$.

In Došlić's proof of this theorem seven cases were enumerated in figure 5 in [5]. But two additional cases that may happen were not listed (See figure 1).

In this note we avoid such an enumeration and give a simplified proof to theorem 3 by mainly applying Euler's formula. For a graph G, we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. For any $V' \subseteq V(G)$ and $E' \subseteq E(G)$, the induced subgraphs by V' and by E' are denoted by $G[V']$ and $G[E']$, respectively. For the sake of convenience, we call an edge cut of a connected graph G *cyclical edge cut* if the deletion of it separates G into two components, each containing a cycle. The other terminology and notation in graph theory used but unexplained in this note are standard and can be found in [1, 9].

First we give the following lemma as a preparation for our proof to theorem 3.

Lemma 4. Let G be a 3-regular, 3-connected plane graph with $c\lambda(G) = k$. Then for every cyclical edge cut $E_0 = \{e_1, e_2, \ldots, e_k\}$, there exist two cycles C' and C'' on distinct components of $G - E_0$, respectively, such that every edge e_i of E_0 has one endpoint on C' and the other one on C''. Furthermore, E_0 is a matching of G.

Proof. Let $E_0 = \{e_1, e_2, \ldots, e_k\}$ be a cyclical edge cut of G. Then E_0 separates G into two components G' and G'' that each contains a cycle, denoted by C_1 and C_2, respectively. We may suppose that the outer face of G is exactly the

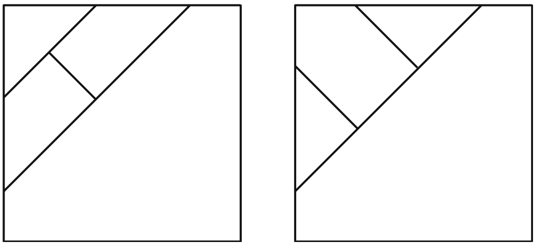

Figure 1. Two possible cases unconsidered in [5].

outer face of G''. Thus G' must lie in some inner face F_0 of G''. We denote the boundaries of F_0 and the outer face of G' by C'' and C', respectively.

We assert that G' and G'' are both 2-connected. If G' is not 2-connected, then there is a cut vertex v of G'. Since $d_{G'}(v) \leqslant 3$, there must exist a cut edge e of G' incident with v. Let us denote the two components of $G' - e$ by G'_1 and G'_2, respectively. Then the cycle C_1 must be contained in G'_1 or in G'_2, say G'_1. On the other hand, there must be at least two edges of E_0 such that each of them has one endpoint in G'_2. Otherwise we would choose fewer than three vertices the deletion of which separates G into at least two components, contradicting the 3-connectivity of G. So the number of edges of E_0 with one endpoint in G'_1 is at most $k - 2$. Thus these edges together with e form an edge cut E'_0 with size of at most $k - 1$ the deletion of which separates G into two components, G'_1 and $G - G'_1$, where G'_1 contains C_1 and $G - G'_1$ contains C_2. Hence, E'_0 is a cyclical edge cut of G and $c\lambda(G) \leqslant k-1$, a contradiction. So G' is 2-connected. Similarly it can be shown that G'' is 2-connected.

Since every face of a 2-connected plane graph is bounded by a cycle (cf. proposition 4.2.5 in [1]), both C' and C'' are cycles. Then by the planarity of G every e_i of E_0 has one endpoint on C' and the other on C''. Since G is 3-regular, each pair of edges of E_0 have no endpoints in common. □

Proof of theorem 3. Since $4 \leqslant c\lambda(G) \leqslant 5$, it is sufficient to prove that $c\lambda(G) \neq 4$. Suppose, to the contrary, that $c\lambda(G) = 4$. Among all cyclical edge cuts of G with size 4, we choose one, denoted by $E_0 := \{e_1, e_2, e_3, e_4\}$, such that one of the two components of $G - E_0$, say G', has the minimum number of vertices. The other component of $G - E_0$ is denoted by G''. Then by lemma 4 there exist two cycles C' and C'' on G' and G'', respectively, such that every edge e_i $(i = 1, 2, 3, 4)$ has one endpoint v'_i on C' and the other endpoint v''_i on C'', and E_0 is a matching of G. From the proof of lemma 4 we can suppose that G' lies in the interior of C'' (See figure 2).

Let us denote the numbers of the additional vertices on C' and C'' by k' and k'', respectively. Then we have that $k' \geqslant 1$ and $k'' \geqslant 1$ since G has no quadrilateral faces.

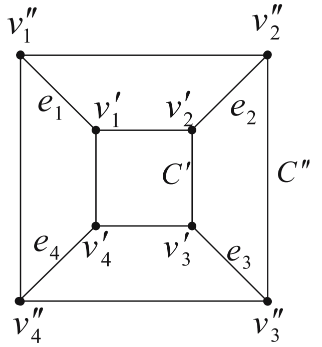

Figure 2. The cycles C' and C'' and the edges connecting them.

Claim. $k' + k'' \leqslant 8$.

In fact, if there are more than eight additional vertices on C' and C'', then there would be at least three additional vertices on one of the boundaries of the four faces of G between C' and C'', resulting in such a face with size more than 6, a contradiction.

Now let us consider the subgraph G' and denote by ν', ϵ' and f' the numbers of vertices, edges and interior faces of G', respectively. Further let r be the number of vertices in the interior of C'. Then we have

$$\nu' = k' + r + 4 \tag{1}$$

and

$$\epsilon' = \frac{8 + 3k' + 3r}{2}. \tag{2}$$

Substituting Equations (1) and (2) into Euler's formula $\nu' - \epsilon' + f' = 1$, we have

$$f' = \epsilon' - \nu' + 1 = \frac{k' + r + 2}{2}. \tag{3}$$

Let m and n denote the numbers of pentagons and the hexagons in the interior of C', respectively. Then we have

$$f' = m + n \tag{4}$$

and

$$\epsilon' = \frac{5m + 6n + k' + 4}{2}. \tag{5}$$

Combining Equations 2–5, we have that

$$m + n = \frac{1}{2}(k' + r + 2),$$
$$5m + 6n = 4 + 2k' + 3r.$$

From the above expressions, we obtain

$$m = k' + 2.$$

So $f' = m + n \geqslant m = k' + 2$; that is, there are at least $k' + 2$ faces of G in the interior of C'. But because of the 3-regularity and the 3-connectivity of G, from each of these k' additional vertices on C' there is exactly one edge towards the interior of C'. So in the interior of C' there must exist at least one face F of G such that the boundary of F is disjoint with C'. Hence, the set, denoted by E', of edges emitted from the k' additional vertices on C' towards the interior of C' is a cyclical edge cut of G with size k' the deletion of which separates G into two components, one of them, denoted by G^*, containing the face F and the other of them containing C'.

Then $|E'| = k' \geqslant 4$ since $c\lambda(G) \geqslant 4$. Applying the same reason on C'', we also have that $k'' \geqslant 4$. Then from the above claim that $k' + k'' \leqslant 8$, we have that $k' = k'' = 4$. So the size of the cyclical edge cut E' is 4. But now the component G^* of $G - E'$ has fewer vertices than G', contradicting our choice that G' has the minimum number of vertices. Hence $c\lambda(G) = 5$. □

By using the cyclical 5-edge-connectivity of fullerene graphs, we now determine their cyclical connectivity. A graph G is *cyclically k-connected* [10] if whenever we can express G as $G = G_1 \cup G_2$, where $E(G_1) \cap E(G_2) = \varnothing$ and G_1 and G_2 both contain cycles, we must have $|V(G_1) \cap V(G_2)| \geqslant k$. The maximum integer k (if exist) such that G is cyclically k-connected is said to be *the cyclical connectivity* of G, denoted by $c\kappa(G)$.

For every fullerene graph G, $c\kappa(G) \leqslant 5$. In fact, let us take a pentagon in G as G_1 and take $G_2 = G - E(G_1)$. Clearly, both G_1 and G_2 contain cycles, $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \varnothing$ and $|V(G_1) \cap V(G_2)| = 5$.

Theorem 5. Let G be a fullerene graph. Then $c\kappa(G) = 5$.

Proof. Because $c\kappa(G) \leqslant 5$, it needs only to prove that equality holds. Suppose, to the contrary, that $c\kappa(G) < 5$. Then there would exist a pair of subgraphs G_1 and G_2 of G such that G_1 and G_2 both contain cycles, $G = G_1 \cup G_2$ and $E(G_1) \cap E(G_2) = \varnothing$, but $|V(G_1) \cap V(G_2)| \leqslant 4$. Among all of such pairs of subgraphs of G, we select a pair of G_1 and G_2 such that $X := V(G_1) \cap V(G_2)$ has as few vertices as possible. Obviously, $|X| \geqslant 3$ since G is 3-connected.

Since G is 3-regular and $E(G_1) \cap E(G_2) = \varnothing$, every vertex $v \in X$ is incident with two edges in one of G_1 and G_2, and one edge (a pendant edge) in the other one. Otherwise, v would be an isolated vertex of G_1 or G_2, say G_1. Let $G_1' = G_1 - v$. Then $G = G_1' \cup G_2$, $E(G_1') \cap E(G_2) = \varnothing$ and G_1' and G_2 both contain cycles, but $|V(G_1') \cap V(G_2)| < |X|$, contradicting the selection of G_1 and G_2. Now let E_0 consist of such pendant edges of G_1 or G_2 each of which is incident with a vertex of X. Then $|E_0| \leqslant |X| \leqslant 4$ and the deletion of E_0 does not destroy the cycles in G_1 and G_2. On the other hand, let $V_1' = V(G_1)\backslash\{v \in X | d_{G_2}(v) = 2\}$ and $V_2' = V(G_2)\backslash\{v \in X | d_{G_1}(v) = 2\}$. Then $V(G) = V_1' \cup V_2'$ and $V_1' \cap V_2' = \varnothing$. Further, it can be seen that every edge between V_1' and V_2' must belong to E_0. In fact, given any edge e with one endpoint v_1 in V_1' and the other one v_2 in V_2'. Without lose of generality, suppose that $e \in E(G_1)$. Then $d_{G_1}(v_2) = 1$ and $v_2 \in X$, that is, e is a pendant edge of G_1 with an endpoint in X. So by the definition of E_0, e belongs to E_0. Hence, E_0 is an edge cut of G. So there must exist a subset E_0' of E_0 such that E_0' is a cyclical edge cut of G with size at most 4. This contradicts the result that $c\lambda(G) = 5$, and the proof is thus finished. □

Došlić in [3] ever gave an alternative definition on cyclical connectivity: a graph G is *cyclically* k*-connected* if it cannot be separated into components of which at least two have cycles, by removing fewer than k vertices. The greatest integer k (if exist) such that G is cyclically k-connected is called *Došlić's cyclical connectivity* of G, denoted by $c\kappa'(G)$.

Došlić showed that $c\kappa'(G) \geqslant 4$ for every fullerene graph G (See Corollary 13 [3]). For a general graph G, here we give a relation between $c\kappa(G)$ and $c\kappa'(G)$.

Theorem 6. $c\kappa(G) \leqslant c\kappa'(G)$.

Proof. If there is no subset of $V(G)$ the deletion of which separates G into components of which at least two have cycles, it is trivial; Otherwise, let us choose a subset X of $V(G)$ with size $c\kappa'(G)$ such that $G-X$ is not connected and at least two components of $G - X$, say G' and G'', respectively, contain cycles. Let $G_1 := G[V(G') \cup X]$ and $G_2 := G[V\backslash V(G')] - E(G[X])$. Then we have that $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \varnothing$, both of G_1 and G_2 have cycles and $V(G_1) \cap V(G_2) = X$. So $c\kappa(G) \leqslant |X|$, i.e., $c\kappa(G) \leqslant c\kappa'(G)$. □

Equation in theorem 6 does not necessarily hold. For example, in figure 3 the graph G is the union of the graph G_1 and G_2, where $V(G_1)\cap V(G_2) = \{x, y\}$. It is seen that $c\kappa(G) = 2$. But there is no any subset of $V(G)$ whose removal from G can separate G into components of which at least two have cycles.

For fullerene graphs, however, we have

Corollary 7. For every fullerene graph G, $c\kappa(G) = c\kappa'(G) = 5$.

Proof. By theorems 5 and 6, it is sufficient to show that $c\kappa'(G) \leqslant 5$. Take a pentagon H in G and let X be the subset of $V(G)$ consisting of the five vertices of $G - V(H)$ each of which is adjacent with a vertex on H. Then the subgraph $G-X$ of G has two components H and $G-X \cup V(H)$, both of them containing cycles. So $c\kappa'(G) \leqslant |X| = 5$. □

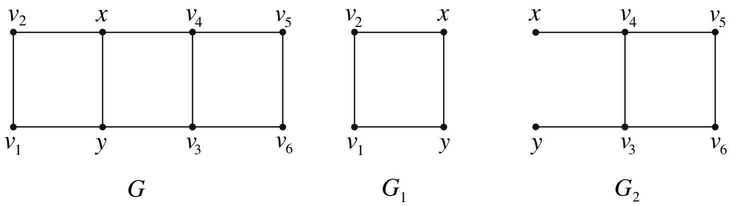

Figure 3. Graph G with two subgraphs G_1 and G_2.

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